MTH 605 - Quiz 1 Solutions

1. Let $p : \widetilde{X} \to X$ be a covering space, where $\tilde{X}$ is path-connected. If there exists $x_0 \in X$ and $k \in \mathbb{N}$ such that $|p^{-1}(x_0)| = k$, then show that $|p^{-1}(x)| = k$ for each $x \in X$.

   Solution. Suppose we assume that $\tilde{X}$ is path-connected with a base-point $\tilde{x}_0 \in p^{-1}(x_0)$. Then by the lifting correspondence, there exists a surjective map $\phi : \pi_1(X, x_0) \to p^{-1}(x_0)$, defined by $\phi([\beta]) = \tilde{\beta}(1)$, where $\tilde{\beta}$ is the unique lift of the loop β beginning at $\tilde{x}_0$. Let x_1 be another point in X distinct from x_0. Since $\tilde{X}$ is path-connected, so is X, and hence there exists an isomorphism $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_0)$, where α is a path in X from x_0 to x_1. As α lifts to a unique path $\tilde{\alpha}$ beginning at $\tilde{x}_0$, we have the following commutative diagram:

   $$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{\hat{\alpha}} & \pi_1(X, x_1) \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ p^{-1}(x_0) & \overset{f_\alpha}{\dashrightarrow} & p^{-1}(x_1), \end{array}$$

   where $f_\alpha := \phi \circ \hat{\alpha} \circ \phi^{-1}$, is a well-defined surjective map (why?) induced by α. Hence, it follows that $|p^{-1}(x_1)| < \infty$. Finally by switching the roles of x_0 and x_1 and arguing as above, we also get a surjective map $p^{-1}(x_1) \to p^{-1}(x_0)$. Therefore, it follows that $|p^{-1}(x_0)| = |p^{-1}(x_1)|$.

2. Let X be complement of the union of n mutually disjoint lines in $\mathbb{R}^3$ that are perpendicular to the xy-plane. Then show that X is homotopically equivalent to the wedge of n-circles. (Note that the wedge of n circles is n distinct copies of S^1 joined at a single point.)

   Solution. Let the mutually disjoint lines be $\ell_1, \ldots, \ell_n$. First, the homotopy $\mathbb{R}^3 \times I \to \mathbb{R}^3$ defined by $H((x, y, z), t) = (x, y, (1-t)z)$ gives a deformation retraction of $\mathbb{R}^3 \setminus \ell_1 \cup \ell_2 \cup \ldots \ell_n$ onto $\mathbb{R}^3 \setminus \{x_1, x_2, \ldots, x_n\}$, where x_i is the point of intersection of ℓ_i with the xy-plane. Next, we deformation retract $\mathbb{R}^3 \setminus \{x_1, x_2, \ldots, x_n\}$ onto $D \setminus \{x_1, x_2, \ldots, x_n\}$, where D is a closed disk in $\mathbb{R}^3$ enclosing $\{x_1, x_2, \ldots, x_n\}$. Then we deformation retract $D \setminus \{x_1, x_2, \ldots, x_n\}$ onto $D_1 \cup D_2 \cup \ldots \cup D_n \setminus \{x_1, x_2, \ldots, x_n\}$, where each D_i is chosen to be a closed disk in the interior of D with center x_i such that $D_i \cap D_{i+1} = \{y_i\}$, for $1 \leq i \leq n-1$ and $D_i \cap D_j = \emptyset$ when $|i - j| > 1$. Next, we simultaneously deformation retract each $D_i \setminus \{x_i\}$ onto its boundary $S_i := \partial D_i$ to obtain a union of circles $C_1 \cup C_2 \cup \ldots \cup C_n$, where $C_i \cap C_{i+1} = \{y_i\}$, for $1 \leq i \leq n-1$ and $C_i \cap C_j = \emptyset$ when $|i - j| > 1$. Then, we consider a path $\alpha = \beta_1 * \beta_2 * \ldots \beta_{n-2}$

in $C_1 \cup C_2 \cup \ldots \cup C_n$ from y_1 to y_{n-1}, where each β_i is one of the subarcs in the circle S_i from y_i to y_{i+1}. Finally, by collapsing the path α in $C_1 \cup C_2 \cup \ldots \cup C_n$, we obtain an induced quotient map $C_1 \cup C_2 \cup \ldots \cup C_n \to \bigvee_{i=1}^{n} S^1$, which is a homotopy equivalence (why?).